Universiti Tun Hussein Onn Malaysia

UNIVERSITI TUN HUSSEIN ONN MALAYSIA

FINAL EXAMINATION
SEMESTER II
SESSION 2014/2015

COURSE NAME	:	MATHEMATICAL MODELLING I
COURSE CODE	:	BWA 30603
PROGRAMME	:	3 BWA
EXAMINATION DATE	:	JUNE 2015/JULY 2015
DURATION	:	3 HOURS
INSTRUCTION	:	ANSWER **ALL** QUESTIONS

THIS QUESTION PAPER CONSISTS OF **FIVE (5)** PAGES

Q1 (a) List the steps that need to be considered when developing a mathematical model.

(3 marks)

(b) Consider a compound interest formula

$$S_n = S_0\left(1+\frac{I}{r}\right)^{nr},$$

where S_n is the future value, S_0 is the principal investment, I is the interest rate, r is the frequency of compounding every year and n is the number of year. There are two options to earn interest. Company A offers compound interest at a 4% rate with a conversion period of one month and Company B offers compound interest at a 6% rate with a conversion period of three months. Which interest offer maximizes the amount on deposit for any value of principal investment after 5 years?

(6 marks)

(c) A plant-herbivore model is written as

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - \frac{aN^2P}{b+N^2},$$
$$\frac{dP}{dt} = \frac{eaN^2P}{b+N^2} - mP,$$

where r, K, a, b and e are positive constants, N is the density of herbivore (scale bugs) and P is the chemical state of the plant. Prove that by using the scaled variables

$$N=\sqrt{b}\,x, P=\sqrt{b}\,y, \quad \text{and} \quad t=\frac{K\tau}{r\sqrt{b}},$$

the model is reduced to the non-dimensional model

$$\frac{dx}{d\tau} = x(\alpha - x) - \frac{\beta x^2 y}{1+x^2},$$
$$\frac{dy}{d\tau} = \frac{\beta_1 x^2 y}{1+x^2} - \gamma y.$$

Determine α, β, β_1 and γ in terms of original parameters.

(16 marks)

Q2 (a) A population is modelled by the equation

$$\frac{dN}{dt} = kN, \quad N(0) = N_0.$$

Find the solution and explain the behaviour of the population for $k < 0$ and $k > 0$.

(5 marks)

(b) Explain each term in the population growth model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - hN + gN,$$

and suggest possible meanings for parameters r, K, h and g.

(4 marks)

(c) Suppose a two-species competition model is written as

$$\frac{dN_1}{dt} = N_1(1 - N_1 - \alpha N_2),$$
$$\frac{dN_2}{dt} = N_2(\beta - N_2 - \gamma N_1).$$

(i) Prove that there are four steady states (N_1^*, N_2^*):

$$(0,0), \quad (0,\beta), \quad (1,0) \quad \text{and} \quad \left(\frac{\alpha\beta - 1}{\alpha\gamma - 1}, \frac{\gamma - \beta}{\alpha\gamma - 1}\right).$$

(7 marks)

(ii) If $\alpha = \beta = \frac{1}{2}, \gamma = \frac{1}{3}$, find the Jacobian matrix of the system.

(2 marks)

(iii) Calculate determinants and traces for each steady-state. Using these information, state whether each steady-state is **stable** or **unstable** without finding the eigenvalues.

(7 marks)

Q3 (a) Define the differences between pure monopoly and pure competition markets.

(4 marks)

(b) Given a linear two-commodity model

$$\begin{aligned} Q_{d_1} - Q_{s_1} &= 0, \\ Q_{d_1} &= a_0 + a_1P_1 + a_2P_2, \\ Q_{s_1} &= b_0 + b_1P_1 + b_2P_2, \\ Q_{d_2} - Q_{s_2} &= 0, \\ Q_{d_2} &= \alpha_0 + \alpha_1P_1 + \alpha_2P_2, \\ Q_{s_2} &= \beta_0 + \beta_1P_1 + \beta_2P_2, \end{aligned}$$

where Q_{d_i} and Q_{s_i} are the quantity demanded of the commodity 1 and quantity of the supplied of the commodity 2, respectively. The price of the commodity 1 is denoted by P_1 and the price of commodity 2 is denoted by P_2.

(i) Reduce the model from six equations to two equations.

(2 marks)

(ii) If $c_i = a_i - b_i$ and $\gamma_i = \alpha_i - \beta_i$, show that the price equilibria are

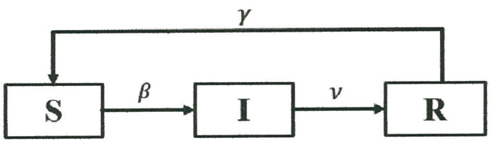
$$P_1 = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1} \quad \text{and} \quad P_2 = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}.$$

(16 marks)

(iii) What is the restriction should be imposed on the model so that the model makes sense.

(3 marks)

Q4 Given an SIRS epidemic model



where S is susceptible, I is infective and R is recovered populations. Parameters β, ν and γ are positive constants. A mathematical formulation for the model is given as

$$\frac{dS}{dt} = -\beta SI + \gamma R,$$
$$\frac{dI}{dt} = \beta SI - \nu I,$$
$$\frac{dI}{dt} = \nu I - \gamma R.$$

The total population is $N = S + I + R$.

(a) Reduce the model to a system that involves only S and I.
(2 marks)

(b) Prove there are two steady-states (S^*, I^*) :

$$(N, 0) \quad \text{and} \quad \left(\frac{\nu}{\beta}, \frac{\gamma(N - \nu/\beta)}{\nu + \gamma}\right).$$

(7 marks)

(c) Find the eigenvalues of each steady-state and classify their behaviours.
(16 marks)

- END OF QUESTION -