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UNIVERSITI TUN HUSSEIN ONN MALAYSIA

FINAL EXAMINATION
SEMESTER II
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DURATION	:	3 HOURS
INSTRUCTION	:	ANSWER **ALL** QUESTIONS

THIS QUESTION PAPER CONSISTS OF **FIVE (5)** PAGES

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Q1 Consider the quadratic problem

$$\text{minimize } f(x) = \frac{1}{2}x^{\mathrm{T}}Qx - b^{\mathrm{T}}x$$

where $x \in \Re^n$ and Q is an $n \times n$ symmetric positive definite matrix. The best direction of search is in the Q-conjugate direction. Basically, two directions $d^{(1)}$ and $d^{(2)}$ in $\Re^n$ are said to be Q-conjugate if $d^{(1)\mathrm{T}}Qd^{(2)} = 0$. In general, we have the following definition:

Definition: *Let Q be a real symmetric $n \times n$ matrix. The directions $d^{(0)}, d^{(1)}, d^{(2)}, \ldots, d^{(m)}$ are Q-conjugate if for all $i \neq j$, we have $d^{(i)\mathrm{T}}Qd^{(j)} = 0$.*

(a) Prove the following proposition.

Proposition: *Let Q be a real symmetric $n \times n$ matrix. If the directions $d^{(0)}, d^{(1)}, \ldots, d^{(k)} \in \Re^n$, $k \leq n-1$, are nonzero and Q-conjugate, then they are linearly independent.*

(12 marks)

(b) Let

$$Q = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Note that $Q = Q^{\mathrm{T}} > 0$. The matrix Q is positive definite because all its leading principal minors are positive:

$$\Delta_1 = 3 > 0,\ \Delta_2 = \det\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = 12 > 0,\ \Delta_3 = \det Q = 20 > 0.$$

Construct a set of Q-conjugate vectors $d^{(0)}, d^{(1)}, d^{(2)}$, where

$$d^{(0)} = (1,0,0)^{\mathrm{T}},\quad d^{(1)} = \left(d_1^{(1)}, d_2^{(1)}, d_3^{(1)}\right),\quad d^{(2)} = \left(d_1^{(2)}, d_2^{(2)}, d_3^{(2)}\right).$$

(13 marks)

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Q2 The conjugate direction algorithm for minimizing the quadratic function

$$f(x) = \frac{1}{2}x^{\mathrm{T}}Qx - b^{\mathrm{T}}x$$

where $Q = Q^{\mathrm{T}} > 0$, $x \in \Re^n$, is given by

Basic conjugate direction algorithm: *Given a starting point $x^{(0)}$ and Q-conjugate directions $d^{(0)}, d^{(1)}, d^{(2)}, \ldots, d^{(n-1)}$, for $k \geq 0$,*

$$g^{(k)} = \nabla f(x^{(k)}) = Qx^{(k)} - b, \qquad \alpha_k = -\frac{g^{(k)\mathrm{T}}d^{(k)}}{d^{(k)\mathrm{T}}Qd^{(k)}}, \qquad x^{(k+1)} = x^{(k)} + \alpha_k d^{(k)}.$$

The following theorem shows the conjugate direction method.

Theorem: *For any starting point $x^{(0)}$, the basic conjugate direction algorithm converges to the unique x^* that solves $Qx = b$ in n steps, that is, $x^{(n)} = x^*$.*

(a) Consider $x^* - x^{(0)} \in \Re^n$. Because the directions $d^{(i)}$, for $i = 0,1,\ldots,n-1$, are linearly independent, there exist constants β_i, for $i = 0,1,\ldots,n-1$, such that

$$x^* - x^{(0)} = \beta_0 d^{(0)} + \cdots + \beta_{n-1}d^{(n-1)}.$$

Show that

$$\beta_k = -\frac{g^{(k)\mathrm{T}}d^{(k)}}{d^{(k)\mathrm{T}}Qd^{(k)}} = \alpha_k$$

and

$$x^* = x^{(n)}.$$

(15 marks)

(b) Find the minimum of

$$f(x_1, x_2) = \frac{1}{2}x^{\mathrm{T}}\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}x - x^{\mathrm{T}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad x \in \Re^2,$$

using the conjugate direction method with the initial point $x^{(0)} = (0,0)^{\mathrm{T}}$, and Q-conjugate directions $d^{(0)} = (1,0)^{\mathrm{T}}$ and $d^{(1)} = (-\frac{3}{8}, \frac{3}{4})^{\mathrm{T}}$.

(10 marks)

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Q3 (a) The minimizer for the quadratic approximation is used as the starting point for the next iteration. This leads to Newton's recursive algorithm

$$x^{(k+1)} = x^{(k)} - F(x^{(k)})^{-1} g^{(k)}.$$

We may try to guarantee that the algorithm has the descent property by modifying the original algorithm as follows:

$$x^{(k+1)} = x^{(k)} - \alpha_k F(x^{(k)})^{-1} g^{(k)},$$

where α_k is chosen to ensure that

$$f(x^{(k+1)}) < f(x^{(k)}).$$

To avoid the computation of $F(x^{(k)})^{-1}$, the quasi-Newton methods use an approximation to $F(x^{(k)})^{-1}$ in place of the true inverse. Consider the formula

$$x^{(k+1)} = x^{(k)} - \alpha H_k g^{(k)},$$

where H_k is an $n \times n$ real symmetric positive definite matrix and α is a positive search parameter. Show that to guarantee a decrease in f for small α, we have

$$g^{(k)T} H_k g^{(k)} > 0.$$

(10 marks)

(b) Quasi-Newton algorithms have the form

$$d^{(k)} = -H_k g^{(k)},$$

$$\alpha_k = \arg\min_{\alpha \geq 0} f(x^{(k)} + \alpha d^{(k)}),$$

$$x^{(k+1)} = x^{(k)} + \alpha_k d^{(k)},$$

where the matrices $H_0, H_1, \ldots$ are symmetric. In the quadratic case, these matrices are required to satisfy

$$H_{k+1} \Delta g^{(i)} = \Delta x^{(i)}, \quad 0 \leq i \leq k,$$

where

$$\Delta x^{(i)} = x^{(i+1)} - x^{(i)} = \alpha_i d^{(i)} \quad \text{and} \quad \Delta g^{(i)} = g^{(i+1)} - g^{(i)} = Q \Delta x^{(i)}.$$

It turns out that quasi-Newton methods are also conjugate direction methods, as stated in the following theorem.

Theorem: *Consider a quasi-Newton algorithm applied to a quadratic function with Hessian $Q = Q^{\mathrm{T}}$ such that for $0 \leq k < n-1$,*

$$H_{k+1} \Delta g^{(i)} = \Delta x^{(i)}, \quad 0 \leq i \leq k,$$

where $H_{k+1} = H_{k+1}^{\mathrm{T}}$. If $\alpha_i \neq 0$, $0 \leq i \leq k$, then $d^{(0)}, \ldots, d^{(k+1)}$ are Q-conjugate.

Prove the theorem above.

(15 marks)

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Q4 In the rank one correction formula, the correction term is symmetric and has the form $a_k z^{(k)} z^{(k)\mathrm{T}}$, where $a_k \in \Re$ and $z^{(k)} \in \Re^n$. Therefore, the update equation is

$$H_{k+1} = H_k + a_k z^{(k)} z^{(k)\mathrm{T}}.$$

Note that

$$\operatorname{rank} z^{(k)} z^{(k)\mathrm{T}} = \operatorname{rank}\left(\begin{bmatrix} z_1^{(k)} \\ \vdots \\ z_n^{(k)} \end{bmatrix} \begin{bmatrix} z_1^{(k)} \cdots z_n^{(k)} \end{bmatrix}\right) = 1$$

and hence the name rank one correction.

(a) Consider the condition $H_{k+1}\Delta g^{(k)} = \Delta x^{(k)}$, show that

$$H_{k+1} = H_k + \frac{(\Delta x^{(k)} - H_k \Delta g^{(k)})(\Delta x^{(k)} - H_k \Delta g^{(k)})^{\mathrm{T}}}{\Delta g^{(k)\mathrm{T}}(\Delta x^{(k)} - H_k \Delta g^{(k)})},$$

with given H_k, $\Delta g^{(k)}$, and $\Delta x^{(k)}$.

(10 marks)

(b) Let

$$f(x_1, x_2) = x_1^2 + \frac{1}{2} x_2^2 + 3.$$

Apply the rank one correction algorithm to minimize f. Use $x^{(0)} = (1,2)^{\mathrm{T}}$ and $H_0 = I_2$.

(15 marks)

- END OF QUESTION -