UNIVERSITI TUN HUSSEIN ONN MALAYSIA

FINAL EXAMINATION
SEMESTER I
SESSION 2022/2023

COURSE NAME	:	NUMERICAL METHODS
COURSE CODE	:	BWA 21503
PROGRAMME CODE	:	BWA
EXAMINATION DATE	:	FEBRUARY 2023
DURATION	:	3 HOURS
INSTRUCTIONS	:	1. ANSWER ALL QUESTIONS 2. THIS FINAL EXAMINATION IS CONDUCTED VIA ☐ Open book ☒ Closed book 3. STUDENTS ARE **PROHIBITED** TO CONSULT THEIR OWN MATERIAL OR ANY EXTERNAL RESOURCES DURING THE EXAMINATION CONDUCTED VIA CLOSED BOOK

THIS QUESTION PAPER CONSISTS OF **SIX (6)** PAGES

Q1 The velocity v of a falling parachutist is given by

$$v = \frac{g\,m}{c}\left(1 - e^{-(c/m)t}\right),$$

where

$g = 9.8\,\text{ms}^{-2}$ is the gravitational force, and

$c = 15\ \text{kgs}^{-1}$ is the drag coefficient.

Taking initial guess of the parachutist mass m as between 40 kg to 60 kg, calculate the mass m using false-position method when the velocity of the falling parachutist is $35\,\text{ms}^{-1}$ at time $t = 9\,\text{s}$. Do the iteration until $|f(m_i)| < \varepsilon = 0.0005$.

[8 marks]

Q2 A steady-state concentration of a substance that reacts with first-order kinetics in an axially-dispersed plug-flow reactor can be expressed as

$$D\left(\frac{c_{i-1} - 2c_i + c_{i+1}}{(\Delta x)^2}\right) - U\left(\frac{c_{i+1} - c_{i-1}}{2(\Delta x)}\right) - kc_i = 0,$$

where D is dispersion coefficient (m^2/hr), c_i is concentration at node i (mol/L), x is distance (m), U is fluid velocity (m/hr) and k is reaction rate per hour. The parameter values for the mass balance problem are $D = 2$, $U = 1$, $k = 0.2$, $\Delta x = 2.5$, $c(x = 0) = 80$ and $c(x = 10) = 20$.

(a) Analyse the above problem by showing that the problem can be written into a system of linear equations as follows:

$$\begin{pmatrix} -0.84 & 0.12 & 0 \\ 0.52 & -0.84 & 0.12 \\ 0 & 0.52 & -0.84 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -41.6 \\ 0 \\ -2.4 \end{pmatrix}.$$

[7 marks]

(b) Hence, solve the system using Gauss-Seidel iteration method with initial concentrations, $c_1 = 54$, $c_2 = 37$ and $c_3 = 26$. Do the iteration until $\max\limits_{1\le i\le n} \{|c_i^{(k+1)} - c_i^{(k)}|\} < \varepsilon = 0.005$.

[7 marks]

Q3 Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & a & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Identify all $a \in \mathbb{R}$ for which

(a) A is invertible (or nonsingular).

[3 marks]

(b) A is strictly diagonally dominant.

[4 marks]

(c) A satisfies one of the symmetric positive definite conditions that is $(a_{ij})^2 < a_{ii}a_{jj}$, $\forall \underset{i \neq j}{i, j} = 1, 2, \ldots, n$.

[3 marks]

Q4 Let $P_3(x)$ be the interpolating polynomial for the data $(0,0)$, $(0.5, y)$, $(1,3)$ and $(2,2)$. Determine the value of y if the coefficient of x^3 in $P_3(x)$ is 6 by using Lagrange polynomial interpolation.

[12 marks]

Q5 For a function f, the Newton divided-difference table is

x_0	$f[x_0]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$
0	0		
		3	
1	?		3
		?	
2	?		

(a) Calculate the missing entries in the table.

[6 marks]

(b) Find the interpolating polynomial $p(x)$.

[2 marks]

Q6 The flow rate of an incompressible fluid in a pipe of radius 1 is given by

$$Q = \int_0^1 2\pi r V \, dr,$$

where r is the distance from centre of the pipe and V is the velocity of the fluid.

(a) Use suitable Simpson's rule to estimate Q if only the following tabulated velocity measurements V as shown in **Table Q6.1** are available.

Table Q6.1 Tabulated velocity measurements

r	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
V	1.0	0.99	0.96	0.91	0.84	0.75	0.64	0.51	0.36	0.19	0.0

[4 marks]

(b) Then, compare your result with the value obtained analytically using $V = 1 - r^2$.

[6 marks]

Q7 By expanding $f(x+h)$ in a Taylor series up to three terms, deduce an expression for the truncation error e^T in the first derivative 2-point forward difference formula,

$$f'(x) = \frac{f(x+h) - f(x)}{h} + e^T .$$

[4 marks]

Q8 Given $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

(a) Show that all the eigenvalues are either positive or negative signs by using Gerschorin's theorem.

[6 marks]

(b) If the dominant eigenvalue is 4.01, compute the smallest eigenvalue and its associated eigenvector by

(i) shifted power method.

(ii) inverse power method.

Start the iteration with $\mathbf{v}^{(0)} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and stop the iteration when $|m_k - m_{k-1}| < 0.05$.

[12 marks]

(c) Analyse the validity of the results in **Q8(b)(i)** and **Q8(b)(ii)**.

[6 marks]

Q9 The following methods can be used to solve an ODE $x' = f(t,x)$. For each method, analyse if it has the following properties: whether it is Taylor series, Second-order Runge-Kutta type or multistep method, and whether it is explicit or implicit method.

(a) $y_{i+1} = y_i + \frac{1}{4}hf(x_i,\ y_i) + \frac{3}{4}hf(x_i + \frac{2}{3}h,\ y_i + \frac{2}{3}k_1)$.

[2 marks]

(b) $y_{i+1} = y_i + \frac{h}{2}\left[3f(x_i, y_i) - f(x_{i-1}, y_{i-1})\right]$.

[2 marks]

(c) $y_{i+1} = y_i + hf\left[x_i + \frac{h}{2},\ y_i + \frac{hf(x_i,\ y_i)}{2}\right]$.

[2 marks]

(d) $y_{i+1} = y_i + hf(x_i, y_i)$.

[2 marks]

(e) $y_{i+1} = y_i + \frac{h}{12}\left[23f(x_i, y_i) - 16f(x_{i-1}, y_{i-1}) + 5f(x_{i-2}, y_{i-2})\right]$.

[2 marks]

- END OF QUESTIONS -

APPENDIX A

Formula:

False Position method $c_i = \dfrac{a_i f(b_i) - b_i f(a_i)}{f(b_i) - f(a_i)}$	Gauss-Seidel iteration method $x_i^{(k+1)} = \dfrac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}}, \ \forall i = 1, 2, 3, .., n$

Lagrange polynomial interpolation

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i), \ \forall i = 1, 2, 3, .., n, \qquad L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

Newton divided-difference polynomial

$$P_n(x) = f_0^{[0]} + f_0^{[1]}(x - x_0) + f_0^{[2]}(x - x_0)(x - x_1) + \ldots + f_0^{[n]}(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[(f_0 + f_n) + 4\sum_{\substack{i=1 \\ i \text{ odd}}}^{n-1} f_i + 2\sum_{\substack{i=2 \\ i \text{ even}}}^{n-2} f_i\right]$$

$$\int_a^b f(x)\,dx \approx \frac{3}{8}h\left[\begin{array}{c}(f_0 + f_n) + 3(f_1 + f_2 + f_4 + f_5 + \cdots + f_{n-2} + f_{n-1}) \\ + 2(f_3 + f_6 + \cdots + f_{n-3})\end{array}\right]$$

2-point forward difference formula $\dfrac{f(x+h) - f(x)}{h}$

Eigenvalues $A\mathbf{v} = \lambda\mathbf{v}$

Gerschorin's theorem

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \qquad D_i = \{z \in \mathbf{C} : |z - a_{ii}| \le r_i\}, \qquad \lambda_k \in \bigcup_{i=1}^{n} D_i \quad \text{for } k = 1, 2, \ldots, n$$

Shifted power method

$$A_{\text{Shifted}} = A - sI, \qquad \lambda_{\text{smallest}} = \lambda_{\text{Shifted}} + s, \qquad \mathbf{v}_n \approx \mathbf{v}_{\text{Shifted}} = \mathbf{v}^{(k+1)}$$

Inverse power method

$$\lambda_{\text{smallest}} = \frac{1}{\lambda_{\text{inverse}}}, \qquad \mathbf{v}_n \approx \mathbf{v}_{\text{Inverse}} = \mathbf{v}^{(k+1)}$$