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Universiti Tun Hussein Onn Malaysia

UNIVERSITI TUN HUSSEIN ONN MALAYSIA

FINAL EXAMINATION
SEMESTER II
SESSION 2022/2023

COURSE NAME	:	MATHEMATICAL MODELLING
COURSE CODE	:	BWA 30803
PROGRAMME CODE	:	BWA
EXAMINATION DATE	:	JULY/AUGUST 2023
DURATION	:	2 HOURS 30 MINUTES
INSTRUCTIONS	:	1. ANSWER ALL QUESTIONS 2. THIS FINAL EXAMINATION IS CONDUCTED VIA ☐ Open book ☒ Closed book 3. STUDENTS ARE **PROHIBITED** TO CONSULT THEIR OWN MATERIAL OR ANY EXTERNAL RESOURCES DURING THE EXAMINATION CONDUCTED VIA CLOSED BOOK

THIS QUESTION PAPER CONSISTS OF **FIVE (5)** PAGES

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Q1 In fluid mechanics, the Reynolds number Re is a dimensionless number involving fluid velocity v, density ρ, viscosity μ and a characteristic length r. Construct the dimensionless product of the variables that represents Reynolds number. (Dimension for ρ is $M\,L^{-3}$, and dimension for μ is $M\,L^{-1}\,T^{-1}$).

(12 marks)

Q2 Consider a simple SIR model that is given by

$$\frac{dS}{dt} = -\beta IS + \gamma R,$$
$$\frac{dI}{dt} = \beta IS - \upsilon I,$$
$$\frac{dR}{dt} = \upsilon I - \gamma R,$$

where β, υ and γ are positive constants and $S + I + R = N$.

(a) Show that the first steady state for the model is $(S_1,\ I_1,\ R_1) = (N, 0, 0)$.

(5 marks)

(b) If $S = \upsilon / \beta$, verify that the second steady state is

$$(S_2,\ I_2,\ R_2) = \left(\frac{\upsilon}{\beta}, \frac{\gamma(\beta N - \upsilon)}{\beta(\upsilon + \gamma)}, \frac{\upsilon(\beta N - \upsilon)}{\beta(\upsilon + \gamma)} \right).$$

(5 marks)

(c) Explain the meaning of steady states $(S_1,\ I_1,\ R_1)$ and $(S_2,\ I_2,\ R_2)$ of the model.

(5 marks)

Q3 Determine the orders of the following difference equations and state if they are linear or non-linear and whether the equation is autonomous or non-autonomous.

(a) $3x_{n+4} + x_n^{\ 2} = x_{n+1}$.

(3 marks)

(b) $2nx_{n-2} + 5x_{n+1} = \sqrt{n}$.

(3 marks)

(c) $3x_{n-1} + 4_{n+1} = \dfrac{1}{n}$.

(3 marks)

Q4 A population has an initial size of 1000 and increases each year by 30% through normal birth and death processes. Each year 450 of the population are harvested and 600 are received into the population through immigration from neighbouring areas.

(a) Deconstruct the difference equation that describes the above scenario. (Make sure you define all variables and explain all terms).

(6 marks)

(b) How many years will it take for the population to exceed 2500 in size?

(7 marks)

Q5 Solve the nonhomogeneous difference equation $x_{n+2} = 6x_{n+1} - 8x_n + 3^n$ with $x_0 = 1$ and $x_1 = 3$.

(9 marks)

Q6 In economics applications, we assume that the price-demand curve as

$$D(n) = -m_d p(n) + b_d, \quad m_d > 0, b_d > 0.$$

The constant m_d represents the sensitivity of consumers to price. We also assume that the price-supply curve relates the supply in any period with the price one period before as

$$S(n+1) = m_s p(n) + b_s, \quad m_s > 0, b_s > 0.$$

The constant m_s is the sensitivity of suppliers to price. A third assumption we make here is that the market price is the price at which the quantity demanded and the quantity supplied are equal, that is when $D(n+1) = S(n+1)$. Thus

$$-m_d p(n) + b_d = m_s p(n) + b_s \quad \text{or}$$
$$p(n+1) = Ap(n) + B = f(p(n))$$

where

$$A = -\frac{m_s}{m_d}, \quad B = \frac{b_d - b_s}{m_d}.$$

Because A is the ratio of the slopes of the supply and demand curves, the ratio determines the behavior of the price sequence. Illustrate three cases of A, that is when $-1 < A < 0$, $A = -1$ and $A < -1$ using cobweb.

(9 marks)

Q7 In a remote region in Malaysia, the dynamics of fly population has been studied and found to satisfy difference equation

$$x_{n+1} = 11 - 0.01x_n^2,$$

where $x_n(>0)$ is the fly population density at generation n.

(a) Determine the fixed point(s) of this model.

(4 marks)

(b) Analyse the stability of the fixed point(s) found in **Q7(a)**.

(6 marks)

(c) Based on your analysis from **Q7(b)**, how will the dynamics evolve as $n \to \infty$ when the initial fly population density is 30?

(3 marks)

- END OF QUESTIONS -

APPENDIX A

FORMULA

First-Order Linear Difference Equations:

Difference equation	Solution
$x_n - ax_{n-1} = 0, \quad n \geq 1$	$x_n = x_0 a^n$
$x_n - x_{n-1} = b(n), \quad n \geq 1$	$x_n = x_0 + \sum_{p=1}^{n} b(p)$
$x_n - ax_{n-1} = b(n)$	$x_n = x_0 a^n + \sum_{p=1}^{n} a^{n-p} b(p)$

Second-Order Linear Difference Equations:

Difference equation: $Ax_n + Bx_{n-1} + Cx_{n-2} = 0$	
Case	**Solution**
Real, unequal roots, $m = p$ and $m = q$	$x_n = K(p)^n + L(q)^n$
Real, roots $m = p$ (repeated)	$x_n = (Kn + L)p^n$
Complex conjugate roots $m = \alpha \pm i\beta$	$x_n = r^n(K\cos n\theta + L\sin n\theta)$ with $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Geometric Digression:

If $S_n = k + kr + kr^2 + \ldots + kr^{n-1}$, then $S_n = \frac{k(r^n - 1)}{r - 1}$ where $r \neq 1$