UNIVERSITI TUN HUSSEIN ONN MALAYSIA

FINAL EXAMINATION
SEMESTER I
SESSION 2019/2020

COURSE NAME	:	OPTIMAL CONTROL
COURSE CODE	:	BWA 31303
PROGRAMME CODE	:	BWA
EXAMINATION DATE	:	DECEMBER 2019 / JANUARY 2020
DURATION	:	3 HOURS
INSTRUCTION	:	ANSWER **ALL** QUESTIONS

THIS QUESTION PAPER CONSISTS OF **SIX (6)** PAGES

Q1 Consider the autonomous pendulum with friction given by

$$\dot{x}_1 = -x_2,$$
$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2 .$$

(a) Give the Jacobian matrix A at the equilibrium point $x = 0$.

(7 marks)

(b) Show that the eigenvalues of A are

$$\lambda_1 = \frac{(-1 + j\sqrt{3})}{2} \text{ and } \lambda_2 = \frac{(-1 - j\sqrt{3})}{2}.$$

(6 marks)

(c) Verify that the Lyapunov function is

$$V(x) = 1.5x_1^2 - x_1x_2 + x_2^2,$$

given that

$$P = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{pmatrix}.$$

(7 marks)

Q2 Consider a linear system

$$\dot{y} = ay + u,$$

with unknown a has the controller

$$u = -ky,$$

where

$$\dot{k} = \gamma y^2,\ \gamma > 0,$$

in which to guarantee the globally stable, that is, $y(t) \to 0,\ t \to \infty$.

(a) Express that the feedback system in the state space is given by

$$\dot{x}_1 = -(x_2 - a)x_1,$$
$$\dot{x}_2 = \gamma x_1^2.$$

(8 marks)

(b) Show that the derivative $\dot{V}(x)$ along the trajectories of the system is

$$\dot{V}(x) = -x_1^2(b-a),$$

given that the Lyapunov function is

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2,\ b > a.$$

(9 marks)

(c) Judge the conditions for the equilibrium point $x = (0,\ b)^{\mathrm{T}}$ to be a globally asymptotically stable.

(3 marks)

Q3 Consider a linear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

has a Lyapunov function

$$V(x) = x^{\mathrm{T}} P x,$$

where

$$P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix},$$

is a real, symmetric positive definite matrix. It is known that the system is asymptotically stable.

(a) Describe that the Lyapunov equation is

$$A^T P + PA = -Q,$$

where Q is a positive definite matrix.

(7 marks)

(b) Obtain the following equations from the Lyapunov equation

$$\begin{aligned} 2p_1 &= -1, \\ -p_1 + 2p_2 &= 0, \\ -2p_2 + 2p_3 &= 1. \end{aligned}$$

(9 marks)

(c) Show that the equations from **Q3 (b)** can be written into a linear system given below

$$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(4 marks)

Q4 Consider the dynamic optimization problem

$$\min_{u} V = \int_{o}^{T} (1+u^2)^{\frac{1}{2}} dt ,$$

subject to

$$\dot{x} = u ,$$
$$x(0) = a, \; a \text{ is given,}$$
$$x(t) \text{ free, } T \text{ is given.}$$

The transversality condition for the terminal line is $\lambda(T) = 0 = \lambda(t)$ for all $t \in [0, T]$.

(a) Define the Hamiltonian function.

(2 marks)

(b) Derive the first-order condition with respect to the control variable and the state variable.

(6 marks)

(c) Show that the control variable and the costate variable are, respectively, given by

$$u = \left(\frac{\lambda^2}{1-\lambda^2} \right)^{\frac{1}{2}} \text{ and } \lambda = k ,$$

where k is a constant.

(8 marks)

(d) Show that the optimal path is a horizontal straight line.

(4 marks)

Q5 The Hamiltonian function for an optimal control problem is defined by

$$H = x^2 + u^2 + \lambda(x+u).$$

The first-order necessary conditions are given as follows

$$2u + \lambda = 0, \ \dot{x} = x + u, \ -\dot{\lambda} = 2x + \lambda.$$

(a) Obtain the control function.

(2 marks)

(b) Express the two-point boundary-value system as follows

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}.$$

(5 marks)

(c) Show that the solutions for the state and the costate are, respectively, given by

$$x(t) = 2\exp\left(\frac{1}{2}t\right) \text{ and } \lambda(t) = \frac{1}{3}\exp(-3t),$$

where the relationship $\lambda = x$ is provided and the initial conditions are

$$x(0) = 2 \text{ and } p(0) = \frac{1}{3}.$$

(13 marks)

– END OF QUESTIONS –